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Global symplectic coordinates on complex domains

Fabrizio Cuccu, Andrea Loi*

Dipartimento di Matematica, Via Ospedale 72, Università di Cagliari, 09100 Cagliari, Italy

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Abstract

In this paper we deal with complex domains $M \subset \mathbb{C}^n$ equipped with a Kähler form $\omega = \frac{i}{2}\partial\bar{\partial}f$, where $f : M \to \mathbb{R}$ only depends on $|z_j|^2$, $j = 1, \ldots, n$ for the complex coordinates $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$. We give an explicit symplectic immersion Φ of (M, ω) into $\mathbb{R}^{2n}$ in Section 2. In Section 3 we study when the map Φ is a global symplectomorphism for the case of complete Reinhardt domains in $\mathbb{C}^2$.

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1. Introduction

Let (M, ω) be a $2n$-dimensional symplectic manifold. By a well-known theorem of Darboux for every point $p \in M$ there exists a neighborhood U of p and a diffeomorphism $\Phi : U \to \mathbb{R}^{2n}$ such that $\Phi^*(\omega_0) = \omega_|$, where $\omega_0 = \sum_{j=1}^n \mathrm{d}x_j \wedge \mathrm{d}y_j$ is the standard symplectic form on $\mathbb{R}^{2n}$ and where $\omega_|$ denotes the restriction of ω to U. In other words one can say that the open set (U, ω) can be equipped with global symplectic coordinates. An interesting question is to understand how large the set U can be taken and, in particular, when

* Corresponding author. Tel.: +39 070 6758527; fax: +39 070 6758511.
E-mail addresses: loi@unica.it (A. Loi); fcuccu@unica.it (F. Cuccu).

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the case $U = M$ occurs, namely when (M, ω) admits global symplectic coordinates. The interest for these kind of questions comes, for example, after Gromov's discovery [8] of the existence of exotic symplectic structures on $\mathbb{R}^{2n}$ (see also [1] for an explicit construction of a four-dimensional symplectic manifold diffeomorphic to $\mathbb{R}^4$ which cannot be symplectically embedded in $(\mathbb{R}^4, \omega_0)$). In [13] McDuff proved the following global version of the Darboux theorem.

Theorem 1.1 (McDuff [13]). *Let (M, ω) be a simply connected and complete n-dimensional Kähler manifold of non-positive sectional curvature. Then there exists a diffeomorphism $\varphi : M \to \mathbb{R}^{2n}$ such that $\varphi^*(\omega_0) = \omega$, being ω_0 the standard symplectic form on $\mathbb{R}^{2n}$.*

See also [3–5] for further properties of McDuff's symplectomorphism $\varphi : M \to \mathbb{R}^{2n}$. In particular, by the previous theorem, the symplectic structure on a Hermitian symmetric space of non-compact type is standard (see also Remark 2.5 for the case of classical bounded domains endowed with their Bergmann forms). More generally, one can study when (M, ω) admits a symplectic immersion into $\mathbb{R}^{2N}$, with $N \geq n$. By a result of Gromov a contractible symplectic manifold always admits an immersion (embedding) in $\mathbb{R}^{2N}$ for N sufficiently large (see [7,9]).

Observe that both McDuff's and Gromov results are existential and the explicit form of the symplectic immersion, embedding or symplectomorphism is, in general, very hard to find. In this paper we find an explicit symplectic immersion

$$\Phi : (M, \omega) \to (\mathbb{R}^{2n}, \omega_0)$$

of (non-compact) domains $M \subset \mathbb{C}^n$ equipped with an exact Kähler form which only depends on $|z_j|^2$, $j = 1, \ldots, n$ for the complex coordinates $(z_1, \ldots, z_n)$ in $\mathbb{C}^n$ (see Lemma 2.1 in Section 2). In Section 3, we study when the map Φ is a global symplectomorphism for the case of complete Reinhardt domains in $\mathbb{C}^2$ and we give several examples.

2. Symplectic coordinates on some domains in $\mathbb{C}^n$

Let $M \subset \mathbb{C}^n$ be a complex domain (open and connected) in $\mathbb{C}^n$ and let ω be a Kähler form on M. Throughout all this paper we will assume that there exists a smooth function $f : M \to \mathbb{R}$ and a smooth function $\tilde{f} : \tilde{M} \to \mathbb{R}$, defined in an open set $\tilde{M} \subset \mathbb{R}^n$ such that $f(z_1, \ldots, z_n) = \tilde{f}(|z_1|^2, \ldots, |z_n|^2)$ and $\omega = \frac{\mathrm{i}}{2}\partial\bar{\partial} f$ (this last condition means that the function f is a Kähler potential for ω). We set $x_j = |z_j|^2$, $j = 1, \ldots, n$ and we denote by $\tilde{f}_{x_j} : \tilde{M} \to \mathbb{R}$ the partial derivatives of $\tilde{f}$ with respect to the x_j-variable and by $\tilde{f}_{x_j x_k} : \tilde{U} \to \mathbb{R}$ the partial derivatives of $\tilde{f}_{x_j}$ with respect to the x_k-variable. Further we denote by $f_{x_j} : M \to \mathbb{R}$ (resp. $f_{x_j x_k} : M \to \mathbb{R}$) the function obtained by first taking the partial derivative of $\tilde{f}$ with respect to x_j (resp. the partial derivatives of $\tilde{f}_{x_j}$ with respect to x_k) and substituting $|z_j|^2 = x_j$ in it, namely $f_{x_j}(z_1, \ldots, z_n) = \tilde{f}_{x_j}(|z_1|^2, \ldots, |z_n|^2)$ ($f_{x_j x_k}(z_1, \ldots, z_n) = \tilde{f}_{x_j x_k}(|z_1|^2, \ldots, |z_n|^2)$ resp.).

The main result of this section is the following lemma.

Lemma 2.1. *Suppose that the functions* $\tilde{f}_{x_k}$ *are strictly positive on* $\tilde{M}$ *for all* $k = 1, \ldots, n$. *Consider the functions on* $\tilde{M}$ *defined by* $\tilde{\varphi}_k = \sqrt{\tilde{f}_{x_k}}$ *for* $k = 1, \ldots, n$ *and set* $\varphi_k(z_1, \ldots, z_n) = \tilde{\varphi}_k(|z_1|^2, \ldots, |z_n|^2)$ *for* $k = 1, \ldots, n$. *Then the map*

$$\Phi : M \to \mathbb{C}^n : z = (z_1, \ldots, z_n) \mapsto (\varphi_1(z)z_1, \ldots, \varphi_n(z)z_n) \tag{1}$$

is a smooth symplectic immersion from (M, ω) *into* $(\mathbb{C}^n = \mathbb{R}^{2n}, \omega_0)$, *i.e.* $\Phi^*(\omega_0) = \omega$, *where*

$$\omega_0 = \frac{\mathrm{i}}{2}\sum_{j=1}^{n} \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_j = \sum_{j=1}^{n} \mathrm{d}\xi_j \wedge \mathrm{d}\eta_j, \quad z_j = \xi_j + i\eta_j$$

is the standard symplectic structure on $\mathbb{R}^{2n}$.

Proof. A straightforward computation using the fact that $\tilde{\varphi}_k^2 = \tilde{f}_{x_k}$ shows that

$$\Phi^*(\omega_0) = \frac{\mathrm{i}}{2}\sum_{l=1}^{n} \mathrm{d}(\varphi_l z_l) \wedge \mathrm{d}(\varphi_l \bar{z}_l)$$
$$= \frac{\mathrm{i}}{2}\sum_{j,k=1}^{n} [\tilde{\varphi}_k^2 \delta_{kj} + (\tilde{\varphi}_k \tilde{\varphi}_{kx_j} + \tilde{\varphi}_j \tilde{\varphi}_{jx_k})\bar{z}_j z_k]\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k,$$

where δ_{kj} is the δ-Kronecker and φ_{kx_j} denote the partial derivative of φ_k with respect to x_j evaluated at $|z_j|^2 = x_j$. On the other hand:

$$\omega = \frac{\mathrm{i}}{2}\sum_{j,k=1}^{n} g_{j\bar{k}}\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k = \frac{\mathrm{i}}{2}\sum_{j,k=1}^{n} (\tilde{f}_{x_k}\delta_{kj} + \tilde{f}_{x_j x_k}\bar{z}_j z_k)\, \mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k. \tag{2}$$

Hence the conclusion follows by

$$f_{x_j x_k} = \varphi_k \varphi_{kx_j} + \varphi_j \varphi_{jx_k} \quad \forall j, k = 1, \ldots, n,$$

which is a consequence of $\tilde{\varphi}_k^2 = \tilde{f}_{x_k}$.

We conclude this section with some examples where, with a slight abuse of notation, we identify the map $\tilde{f}$ with the Kähler potential *f*.

If the manifold *M* has complex dimension 1 then the Kähler potential *f* only depends on one variable, say $x = |z|^2$. Therefore, since $\omega = \frac{\mathrm{i}}{2}\partial\bar{\partial} f$ is a Kähler form, it follows that the function $f_x + x f_{xx}$ is strictly positive and so the function

$$A(x) = x f_x, \quad x = |z|^2$$

is invertible. Denote by $G(x)$ its inverse. Denote by $S \subset \mathbb{C}$ the set of points in $\mathbb{C}$ where the function

$$\psi(z) = \frac{\sqrt{G(|z|^2)}}{|z|}$$

is not defined. Then, by Theorem 3.2, or by a direct computation, one easily obtains the following corollary.

Corollary 2.2. *Suppose that f_x is a strictly positive function then the map*

$$\Phi : M \to \mathbb{C} \setminus \{S\} : z \mapsto \varphi(|z|^2)z, \quad \varphi = \sqrt{f_x}$$

defines a symplectomorphism of (M, ω) to $\mathbb{C} \setminus \{S\}$ endowed with the restriction of the symplectic form $\omega_0 = \mathrm{d}x \wedge \mathrm{d}y$ of $\mathbb{R}^2$ with inverse given by:

$$\Psi : \mathbb{C} \setminus \{S\} \to M : z \mapsto \psi(z)z.$$

Example 2.3. Let $M = \mathbb{C} \setminus \{\overline{\mathcal{D}}_1\} \subset \mathbb{C}$ be the complement of the closed unit disk in $\mathbb{C}$ endowed with the Kähler form

$$\omega = \frac{\mathrm{i}}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{|z|^2} = \frac{\mathrm{i}}{4}\partial\bar{\partial}\log^2|z|^2.$$

The function f is given in this case by:

$$f : (1, +\infty) \to \frac{\log^2(x)}{2},$$

$f_x = \frac{\log x}{x}$ which is strictly positive (since we are assuming that $x = |z|^2 > 1$) and the set S is given by the point $z = 0$. By the previous corollary one has that the map

$$\Phi : M \to \mathbb{C} \setminus \{0\} : z \mapsto \sqrt{\frac{\log|z|^2}{|z|^2}}z$$

is a symplectomorphism with inverse

$$\Psi : \mathbb{C} \setminus \{0\} \to M : z \mapsto \frac{\sqrt{\mathrm{e}^{|z|^2}}}{|z|}z.$$

Example 2.4. Let $M = \mathcal{D}_1 \subset \mathbb{C}$ be the unit disk in $\mathbb{C}$ endowed with the Kähler form

$$\omega = \frac{\mathrm{i}}{2}\frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 - |z|^2)^2} = -\frac{\mathrm{i}}{2}\partial\bar{\partial}\log(1 - |z|^2).$$

The function f is given in this case by:

$$f : (0, 1) \to -\log(1 - x),$$

$f_x = \frac{1}{1-x} > 0$ and S reduces to the empty set. By Corollary 2.2 the map

$$\Phi : \mathcal{D}_1 \to \mathbb{C} = \mathbb{R}^2 : z \mapsto \frac{z}{\sqrt{1 - |z|^2}} \tag{3}$$

is a symplectomorphism with inverse

$$\Psi : \mathbb{C} \to \mathcal{D}_1 : z \mapsto \frac{z}{\sqrt{1 + |z|^2}}.$$

More generally, let $\mathcal{D}_n = \{z \in \mathbb{C}^n \,|\, |z|^2 = \sum_{j=1}^n |z_j|^2 < 1\}$ be the n-dimensional ball in $\mathbb{C}^n$ endowed with the hyperbolic Kähler form $\omega = -\frac{i}{2}\partial\bar{\partial}\log(1 - |z|^2)$. Then, one can easily verify that the map (3) (with $z = (z_1, \dots, z_n)$) defines a global symplectomorphism of $(\mathcal{D}_n, \omega)$ into $(\mathbb{C}^n = \mathbb{R}^{2n}, \omega_0)$.

Remark 2.5. The previous example generalize to the case Hermitian symmetric spaces of non-compact type due to an unpublished work of J. Rawnsley. Here we consider the first Cartan's domain, namely

$$\mathcal{D} = \{Z \in M_{m,n}(\mathbb{C}) | I_m - ZZ^* > 0\}, \quad m, n \in \mathbb{N},$$

endowed with the Kähler form

$$\omega = \frac{i}{2}\partial\bar{\partial}\log K = -(m + n)\frac{i}{2}\partial\bar{\partial}\log \det(I_m - ZZ^*).$$

Here I_m denotes the $m \times m$ identity matrix and $A > 0$ (for a matrix A with real entries) means that A is positive definite (see e.g. [10] for details). Then, one can show that the map:

$$\Phi : \mathcal{D} \to \mathbb{R}^{2nm} : Z \mapsto \sqrt{m + n}(I_m - ZZ^*)^{1/2} Z \tag{4}$$

is a diffeomorphism satisfying $\Phi^*(\omega_0) = \omega$.

3. The case of complete Reinhardt domains

In this section we study the symplectic coordinates on complete Reinhardt domains in $\mathbb{C}^2$. These domains have been extensively studied by several authors in the complex geometry context (see e.g. [6,12,2,11]). Recall that a domain $M \subset \mathbb{C}^2$ is called *Reinhardt* if $z = (z_1, z_2) \in M$ whenever $w = (w_1, w_2) \in M$ and $|z_1| = |w_1|$, $|z_2| = |w_2|$. If the same holds even for all z with $|z_1| \le |w_1|$ and $|z_2| \le |w_2|$, the Reinhardt domain is called *complete*.

One can show that any complete Reinhardt domain is of the form

$$M = \mathcal{D}_F = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 < x_0, |z_2|^2 < F(|z_1|^2)\}, \tag{5}$$

where $F : [0, x_0) \to (0, +\infty]$ is a non-increasing lower semi-continuous function from the interval $[0, x_0) \subset \mathbb{R}$ to the extended positive reals $(0, +\infty]$ (the case $x_0 = +\infty$ is not excluded).

In the hypothesis that $F(0) < \infty$, one can define a real two-form on $\mathcal{D}_F$ by

$$\omega_F = \frac{\mathrm{i}}{2}\partial\bar{\partial}\log\frac{1}{F(|z_1|^2) - |z_2|^2}. \tag{6}$$

The following proposition gives us the conditions under which ω_F is a Kähler form on $\mathcal{D}_F$.

Proposition 3.1. *Assume that F is continuous on* $[0, x_0)$ *and* C^2 *on* $(0, x_0)$. *The following conditions are equivalent*:

- (i) ω_F *is a Kähler form on* $\mathcal{D}_F$,
- (ii) *the function* $A(x) = -\frac{xF'(x)}{F(x)}$, *satisfies* $A'(x) > 0$ $\forall x \in [0, x_0)$, *where* F' *denotes the first derivative of F with respect to x,*
- (iii) $\mathcal{D}_F$ *is strongly pseudoconvex.*

Proof. For the proof see [6]. We just give here the proof of the equivalence (i) ⇔ (ii) since we will need it later.

Let $\omega_F = \frac{\mathrm{i}}{2}\sum_{j,k=1}^{2} g_{j\bar{k}}\,\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$ be the expression of the Kähler form ω_F in the (global) coordinates (z_1, z_2). A simple calculation shows that

$$g_{1\bar{1}} = \left.\frac{-HF' - HxF'' + xF'^2}{H^2}\right|_{x=|z_1|^2}, \qquad g_{1\bar{2}} = \bar{g}_{2\bar{1}} = \left.\frac{-F'}{H^2}\bar{z}_1 z_2\right|_{x=|z_1|^2},$$

$$g_{2\bar{2}} = \left.\frac{F}{H^2}\right|_{x=|z_1|^2}, \tag{7}$$

where H is the real valued function on $\mathcal{D}_F$ defined by $H(z_1, z_2) = F(|z_1|^2) - |z_2|^2$. An easy calculation shows that:

$$\det\, g_{j\bar{k}} = g_{1\bar{1}}g_{2\bar{2}} - |g_{1\bar{2}}|^2 = -\frac{F^2}{H^3}\left.\left(\frac{xF'}{F}\right)'\right|_{x=|z_1|^2}. \tag{8}$$

The form ω_F satisfy the Kähler condition if and only if the matrix $g_{j\bar{k}}$ is positive definite and, since $g_{2\bar{2}} > 0$, this is the case if and only if $\det\, g_{j\bar{k}} > 0$ which, by (8), turns out to be equivalent to (ii).

In the sequel we will suppose ω_F is a Kähler form. Since we only work in the smooth case we will also assume that F is a smooth function on $[0, x_0)$.

Our Theorem 3.2 shows that every complete Reinhardt domain $(\mathcal{D}_F, \omega_F)$ admits a symplectic embedding into $(\mathbb{C}^2 = \mathbb{R}^4, \omega_0)$ which turns out to be a symplectomorphism if we restrict the codomain to $\mathbb{C}^2 \setminus \{S\}$ for a set $S \subset \mathbb{C}^2$ whose description follows. By (ii) in Proposition 3.1, the function

$$A(x) = -\frac{xF'(x)}{F(x)},$$

defined in $[0, x_0)$ is invertible. We denote by

$$G : [0, F(x_0)) \to [0, x_0) \tag{9}$$

its inverse. Let z_1 and z_2 in $\mathbb{C}$, set $\xi = \frac{|z_1|^2}{1+|z_2|^2}$ and let $S \subset \mathbb{C}^2$ be the set of points in $\mathbb{C}^2$ where the real-valued function

$$\psi_1(z_1, z_2) = \frac{1}{|z_1|}\sqrt{G(\xi)} \tag{10}$$

is not defined.

We can now state and prove our main result on complete Reinhardt domains.

Theorem 3.2. *Let $\mathcal{D}_F$ be complete Reinhardt domain endowed with the Kähler form ω_F given by* Proposition 3.1. *For $(z_1, z_2) \in \mathcal{D}_F$, define*

$$\varphi_1(z_1, z_2) = \sqrt{\frac{-F'(|z_1|^2)}{F(|z_1|^2) - |z_2|^2}}, \qquad \varphi_2(z_1, z_2) = \frac{1}{\sqrt{F(|z_1|^2) - |z_2|^2}}.$$

Then the map

$$\Phi : (\mathcal{D}_F, \omega_F) \to (\mathbb{C}^2 \setminus \{S\}, \omega_0) : (z_1, z_2) \mapsto (\varphi_1 z_1, \varphi_2 z_2), \tag{11}$$

is a symplectomorphism, where we are equipping $\mathbb{C}^2 \setminus \{S\}$ with the restriction of the standard symplectic structure ω_0 on $\mathbb{R}^4 = \mathbb{C}^2$.

Proof. Observe that the Kähler potential for the form ω_F on $\mathcal{D}_F$ is given by $f = -\log(F(|z_1|^2) - |z_2|^2)$ and $f_{x_1} = \varphi_1^2$ and $f_{x_2} = \varphi_2^2$, where $x_j = |z_j|^2$, $j = 1, 2$. Therefore by Lemma 2.1, the map Φ defines an symplectic immersion of $(\mathcal{D}_F, \omega_F)$ into $(\mathbb{C}^2, \omega_0)$. Define the real valued functions

$$\psi_1 = \frac{1}{|z_1|}\sqrt{G(\xi)}, \qquad \psi_2 = \sqrt{\frac{F(G(\xi))}{1 + |z_2|^2}}, \qquad \xi = \frac{|z_1|^2}{1 + |z_2|^2} \tag{12}$$

on $\mathbb{C}^2 \setminus \{S\}$. One can easily verify that the map

$$\Psi : \mathbb{C}^2 \setminus \{S\} \to \mathcal{D}_F : z = (z_1, z_2) \mapsto (\psi_1(z)z_1, \psi_2(z)z_2) \tag{13}$$

is the desired inverse of the map Φ.

Example 3.3. Let F be the real-valued, strictly decreasing smooth function on $[0,+\infty)$ defined by:

$$F:[0,+\infty)\to\mathbb{R}:x\mapsto\frac{c}{x+c},\quad c>0.$$

It defines the complete Reinhardt domain

$$\mathcal{D}_F=\left\{(z_1,z_2)\in\mathbb{C}^2 \,|\, |z_2|^2<\frac{c}{|z_1|^2+c}\right\}.$$

Since

$$\frac{xF'}{F}=-\frac{x}{x+c},\qquad\left(\frac{xF'}{F}\right)'=-\frac{c}{(x+c)^2}<0\quad\forall x\in[0,1)$$

by Proposition 3.1 we get a well-defined Kähler form ω_F on $\mathcal{D}_F$. Moreover the function G given by (9), namely the inverse of the function $A(x)=-\frac{xF'(x)}{F(x)}=\frac{x}{x+c}$, is given by:

$$G(x)=\frac{cx}{1-x}.$$

Consequently

$$\psi_1(z_1,z_2)=\frac{1}{|z_1|}\sqrt{G(\xi)}=\sqrt{\frac{c}{1-|z_1|^2+|z_2|^2}}.$$

Set

$$S=\{(z_1,z_2)\in\mathbb{C}^2|1-|z_1|^2+|z_2|^2\le 0\}.$$

Thus, by Theorem 3.2, $(\mathcal{D}_F,\omega_F)$ is symplectomorphic to $(\mathbb{C}^2\setminus\{S\},\omega_0)$ via the map

$$\Phi:\mathcal{D}_F\to\mathbb{C}^2\setminus\{S\},\qquad(z_1,z_2)\mapsto\left(\left(\frac{c}{(c+|z_1|^2)(c-c|z_2|^2-|z_1|^2|z_2|^2)}\right)^{1/2}z_1,\ \left(\frac{c+|z_1|^2}{(c-c|z_2|^2-|z_1|^2|z_2|^2)}\right)^{1/2}z_2\right).$$

Example 3.4. Let F be the real-valued smooth function on $[0,+\infty)$ defined by:

$$F:[0,+\infty)\to\mathbb{R}:x\mapsto\frac{1}{(x+1)^p},$$

where p is a positive integer. Since $F'(x) = -p(x+1)^{-p-1} < 0$. The function F defines the complete Reinhardt domain

$$\mathcal{D}_F = \left\{ (z_1, z_2) \in \mathbb{C}^2 | |z_2|^2 < \frac{1}{(|z_1|^2+1)^p} \right\}.$$

Moreover

$$\frac{xF'}{F} = -\frac{px}{x+1}, \qquad \left(\frac{xF'}{F}\right)' = -\frac{p}{(x+1)^2} < 0 \quad \forall x \in [0, +\infty)$$

by Proposition 3.1 we get a well-defined Kähler form ω_F on $\mathcal{D}_F$. Moreover the function G given by (9), namely the inverse of the function $A(x) = -\frac{xF'(x)}{F(x)} = -\frac{px}{x+1}$, is given by:

$$G(x) = \frac{x}{p-x}.$$

Consequently

$$\psi_1(z_1, z_2) = \frac{1}{|z_1|}\sqrt{G(\xi)} = \frac{1}{\sqrt{p(1+|z_2|^2) - |z_1|^2}}.$$

Set

$$S = \{(z_1, z_2) \in \mathbb{C}^2 | p(1+|z_2|^2) - |z_1|^2 \leq 0\}.$$

Thus, by Theorem 3.2, $(\mathcal{D}_F, \omega_F)$ is symplectomorphic to $(\mathbb{C}^2 \setminus \{S\}, \omega_0)$ via the map

$$\Phi : \mathcal{D}_F \to \mathbb{C}^2 \setminus \{S\}, \qquad (z_1, z_2) \mapsto \left(\left(\frac{p}{(|z_1|^2+1)^{p+1} - |z_2|^2(|z_1|^2)+1)} \right)^{1/2} z_1, \left(\frac{(|z_1|^2+1)^p}{(|z_1|^2+1)^p - |z_2|^2} \right)^{1/2} z_2 \right).$$

We now give two examples of complete Reinhardt domains $(\mathcal{D}_F, \omega_F)$ namely for which the set S above reduces to the empty set and hence admitting a system of global symplectic coordinates.

Example 3.5. Let F be the real-valued, strictly decreasing smooth function on $[0, 1)$ defined by:

$$F : [0, 1) \to \mathbb{R} : x \mapsto (1-x)^p, \quad p > 0.$$

Its associated complete Reinhardt domain is given by:

$$\mathcal{D}_F = \{z \in \mathbb{C}^2 | |z_1|^2 + |z_2|^{2/p} < 1\}.$$

Since

$$\frac{xF'}{F} = -\frac{px}{1-x}, \qquad \left(\frac{xF'}{F}\right)' = -\frac{p}{(1-x)^2} < 0 \quad \forall x \in [0, 1)$$

by Proposition 3.1 we get a well-defined Kähler form ω_F on $\mathcal{D}_F$. Moreover the function G given by (9), namely the inverse of the function $A(x) = -\frac{xF'(x)}{F(x)} = \frac{px}{1-x}$, is given by:

$$G(x) = \frac{x}{x+p}.$$

Consequently

$$\psi_1(z_1, z_2) = \frac{1}{|z_1|}\sqrt{G(\xi)} = \frac{1}{\sqrt{(|z_1|^2 + p(1+|z_2|^2)}},$$

which is globally defined on $\mathbb{C}^2$. Therefore by Theorem 3.2, $(\mathcal{D}_F, \omega_F)$ is symplectomorphic to $(\mathbb{R}^4, \omega_0)$ via the map

$$\Phi : \mathcal{D}_F \to \mathbb{C}^2 = \mathbb{R}^4,$$

$$(z_1, z_2) \mapsto \left(\left(\frac{p(1-|z_1|^2)^{p-1}}{(1-|z_1|^2)^p - |z_2|^2}\right)^{1/2} z_1, \left(\frac{1}{(1-|z_1|^2)^p - |z_2|^2}\right)^{1/2} z_2\right).$$

Observe that for $p = 1$ our domain is the unitary disk endowed with the hyperbolic metric (cf. Example 2.4).

Example 3.6. Let $F(x) = \mathrm{e}^{-x}$ in the interval $[0, +\infty)$. Since $F'(x) = -\mathrm{e}^{-x} < 0$, the function F defines a complete Reinhardt domain $\mathcal{D}_F$. Further

$$\frac{xF'}{F} = -x, \qquad \left(\frac{xF'}{F}\right)' = -1,$$

and hence, by Proposition 3.1 we get a well-defined Kähler form ω_F on $\mathcal{D}_F$. In this case $G(x) = x$ and

$$\psi_1(z_1, z_2) = \frac{1}{\sqrt{(1+|z_2|^2)}}.$$

Therefore by Theorem 3.2, $(\mathcal{D}_F, \omega_F)$ is symplectomorphic to $(\mathbb{R}^4, \omega_0)$ via the map

$$\Phi : \mathcal{D}_F \to \mathbb{C}^2 = \mathbb{R}^4,$$

$$(z_1, z_2) \mapsto \left(\left(\frac{e^{-|z_1|^2}}{e^{-|z_1|^2} - |z_2|^2} \right)^{1/2} z_1, \left(\frac{1}{e^{-|z_1|^2} - |z_2|^2} \right)^{1/2} z_2 \right).$$

Other examples: Let $F : [0, x_0) \to (0, +\infty)$ be a strictly decreasing smooth function and $A(x) = -\frac{xF'}{F}$ such that $A'(x) > 0$ so defining a complete Reinhardt domain D_F with a Kähler form ω_F. In the following two examples we change the function F in order to build new domains of $\mathbb{C}^2$. In both the examples one has to avoid the points with $z_1 = 0$ either to obtain a well-defined Kähler form (Example 3.7) or to get a well-defined domain (Example 3.9).

Example 3.7. Fix an integer $n > 1$ and consider the function

$$\tilde{F}(x) = F(x^n). \tag{14}$$

Observe that

$$\tilde{A}(x) = -\frac{x\tilde{F}'(x)}{\tilde{F}(x)} = -n\frac{x^n F'(x^n)}{F(x^n)} = nA(x^n),$$

and hence

$$\tilde{A}'(x) = n^2 x^{n-1} A'(x^n) \geq 0,$$

which vanishes for $x = 0$. It follows by Proposition 3.1 that the form

$$\omega_{\tilde{F}} = -\frac{i}{2}\partial\bar{\partial} \log(\tilde{F}(|z_1|^2) - |z_2|^2)$$

defines a Kähler form on

$$\mathcal{D}_{\tilde{F}} = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1| \neq 0, |z_1|^2 < x_0, |z_2|^2 < \tilde{F}(|z_1|^2)\},$$

where we take out the point with $z_1 = 0$ because at these points $\omega_{\tilde{F}}$ is degenerate, namely the corresponding quadratic form is not positive definite. Therefore, if we denote by $\tilde{G}(x)$ the inverse of $\tilde{A}(x)$ in the interval $(0, \tilde{A}(x_0))$ we get:

$$\tilde{G}(x) = \left(G\left(\frac{x}{n}\right) \right)^{1/n}, \tag{15}$$

where $G(x)$ denotes the inverse of $A(x)$.

As for Theorem 3.2 we get the following proposition.

Proposition 3.8. *Let $\tilde{S}$ be the subset of $\mathbb{C}^2$ consisting of points where the real valued function*

$$\tilde{\psi}_1(z_1, z_2) = \frac{1}{|z_1|}\left(G\left(\frac{\xi}{n}\right)\right)^{1/2n}, \quad \xi = \frac{|z_1|^2}{1+|z_2|^2}$$

is not defined. Then the map

$$\tilde{\Phi} : (\mathcal{D}_{\tilde{F}} \setminus \{z_1 = 0\}, \omega_{\tilde{F}}) \to (\mathbb{C}^2 \setminus \{\tilde{S}\}, \omega_0) : (z_1, z_2) \mapsto (\tilde{\varphi}_1 z_1, \tilde{\varphi}_2 z_2),$$
$$\tilde{\varphi}_1(z_1, z_2) = \sqrt{\frac{-\tilde{F}'(|z_1|^2)}{\tilde{F}(|z_1|^2) - |z_2|^2}}, \ \tilde{\varphi}_2(z_1, z_2) = \frac{1}{\sqrt{\tilde{F}(|z_1|^2) - |z_2|^2}} \tag{16}$$

is a symplectomorphism whose inverse is given by:

$$(z_1, z_2) \mapsto \left(\frac{1}{|z_1|}\sqrt{\tilde{G}(\xi)}z_1, \sqrt{\frac{\tilde{F}(\tilde{G}(\xi))}{1+|z_2|^2}}z_2\right), \quad \xi = \frac{|z_1|^2}{1+|z_2|^2}.$$

Example 3.9. Let q be a non-negative real number and consider the function

$$\hat{F}(x) = \frac{F(x)}{x^q}. \tag{17}$$

Consider the domain

$$\mathcal{D}_{\hat{F}} = \{(z_1, z_2) \in \mathbb{C}^2 || z_1| \neq 0, |z_1|^2 < x_0, |z_2|^2 < \hat{F}(|z_1|^2)\}.$$

Observe that

$$\hat{A}(x) = -\frac{x\hat{F}'(x)}{\hat{F}(x)} = -\frac{xF'}{F} + q = A(x) + q.$$

Thus $\hat{A}'(x) = A'(x) > 0$. Thus, it follows by Proposition 3.1 that the two-form

$$\omega_{\hat{F}} = -\frac{\mathrm{i}}{2}\partial\bar{\partial}\log(\hat{F}(|z_1|^2) - |z_2|^2)$$

is a Kähler form on $\mathcal{D}_{\hat{F}}$. Let us denote by $\hat{G}(x)$ the inverse of $\hat{A}(x)$ in the interval $(q, \hat{A}(x_0))$. Therefore,

$$\hat{G}(x) = G(x - q). \tag{18}$$

As for Theorem 3.2 we then get the following proposition.

Proposition 3.10. *Let $\hat{S}$ be the subset of $\mathbb{C}^2$ consisting of points where the real valued function*

$$\hat{\psi}_1(z_1, z_2) = \frac{1}{|z_1|}\sqrt{G(\xi - q)}, \quad \xi = \frac{|z_1|^2}{1 + |z_2|^2}$$

is not defined. Then the map

$$\hat{\Phi} : (\mathcal{D}_{\hat{F}} \setminus \{z_1 = 0\}, \omega_{\hat{F}}) \to (\mathbb{C}^2 \setminus \{\tilde{S}\}, \omega_0) : (z_1, z_2) \mapsto (\hat{\varphi}_1 z_1, \hat{\varphi}_2 z_2),$$

$$\hat{\varphi}_1(z_1, z_2) = \sqrt{\frac{-\hat{F}'(|z_1|^2)}{\hat{F}(|z_1|^2) - |z_2|^2}}, \ \hat{\varphi}_2(z_1, z_2) = \frac{1}{\sqrt{\hat{F}(|z_1|^2) - |z_2|^2}} \tag{19}$$

is a symplectomorphism whose inverse is given by:

$$(z_1, z_2) \mapsto \left(\frac{1}{|z_1|}\sqrt{\hat{G}(\xi)}z_1, \sqrt{\frac{\hat{F}(\hat{G}(\xi))}{1 + |z_2|^2}}z_2\right), \quad \xi = \frac{|z_1|^2}{1 + |z_2|^2}.$$

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